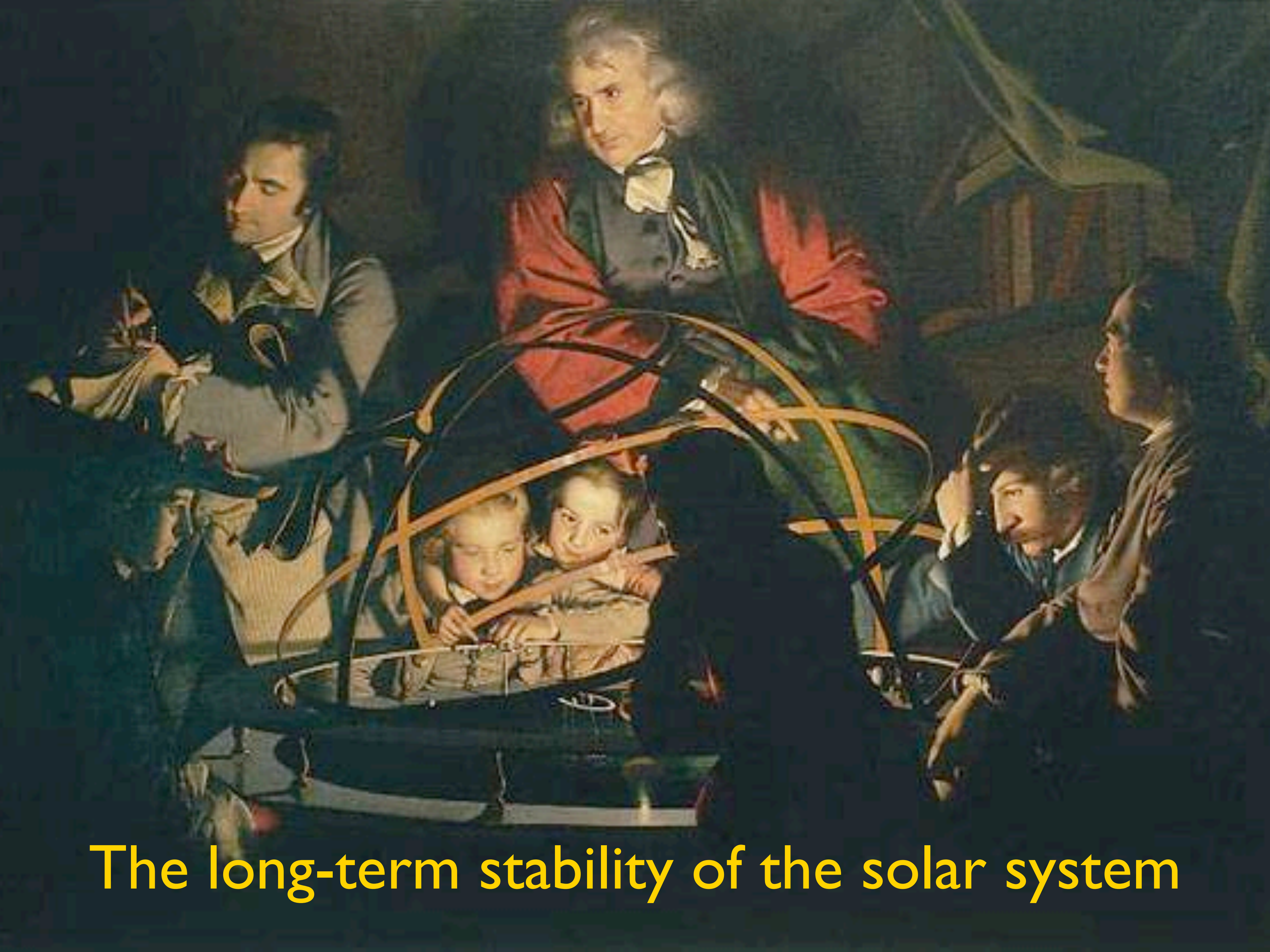

The long-term stability of the solar system

Long-term stability of the solar system

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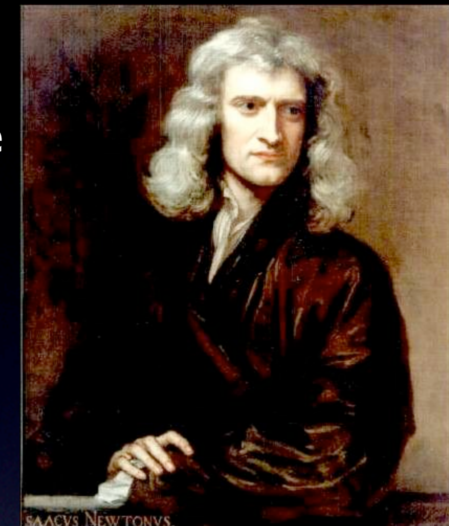
A point mass is surrounded by $N > 1$ much smaller masses on nearly circular, nearly coplanar orbits. Is the configuration stable over very long times (up to 10^{10} orbits)?

Why is this interesting?

- one of the oldest problems in theoretical physics

Newton (1642-1726):

> “blind fate could never make all the planets move one and the same way in orbs concentric, some inconsiderable irregularities excepted, which could have arisen from the mutual actions of planets upon one another, and which will be apt to increase, until this system wants a reformation”



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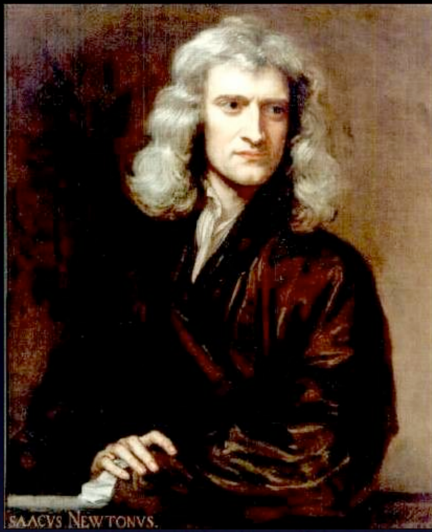

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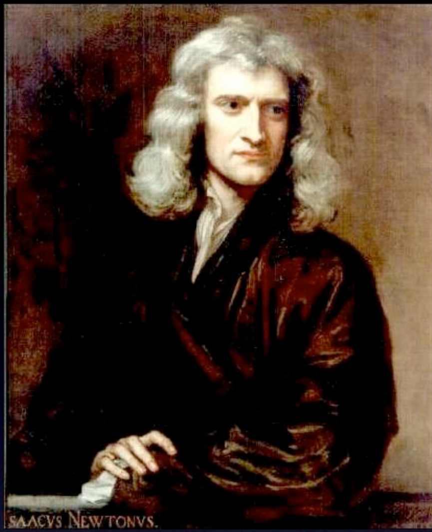

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theism

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causal determinism

Long-term stability of the solar system

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- what is the fate of the Earth?

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four choices:

1. in about 7×10^9 years, the Sun exhausts its fuel and expands into a giant star, heating the Earth to several thousand K and perhaps swallowing it
2. the Earth or some other planet's orbit is unstable, and they collide
3. the Earth's orbit is unstable and it falls into the Sun
4. the Earth's orbit is unstable, and it is ejected into interstellar space

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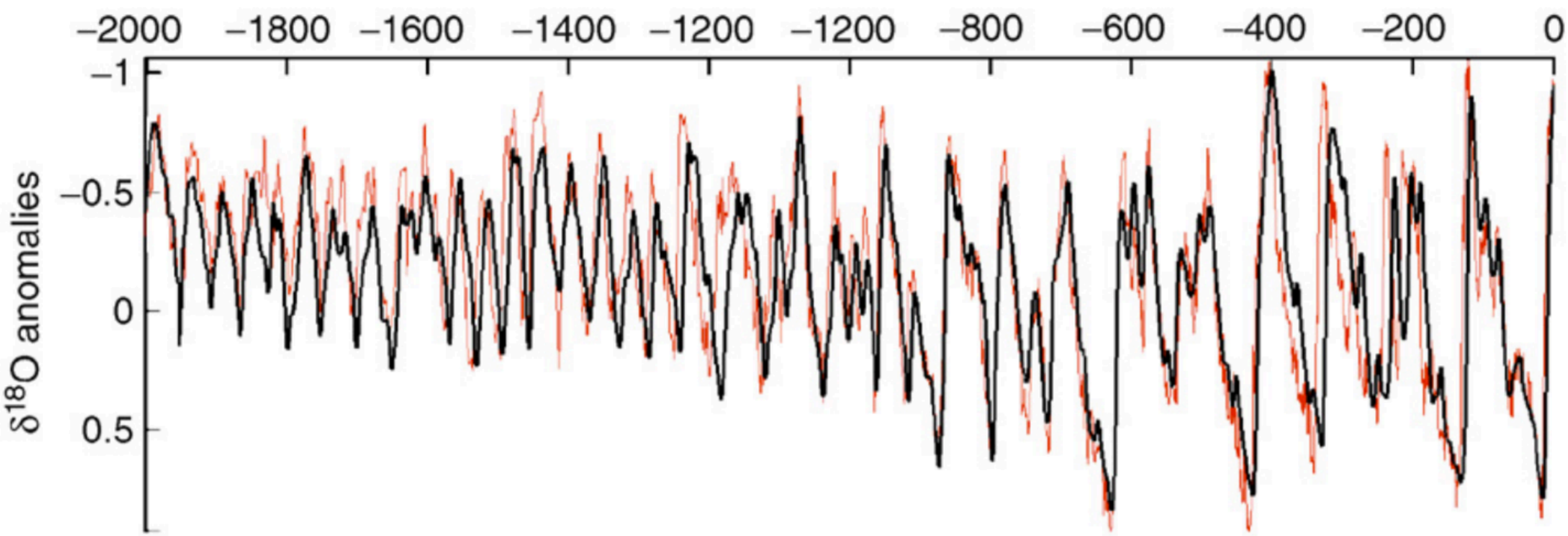
- what is the fate of the Earth?
- why are there so few planets in the solar system?
- can we calibrate the geological timescale over the last 50 Myr?

thousands of years before present



Huybers (2007)

Lisiecki and Raymo (2005)

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- how do dynamical systems behave over very long times?


Large Hadron Collider

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- how do dynamical systems behave over very long times?
- can we explain the properties of extrasolar planetary systems?

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Long-term stability of the solar system

Les personnes qui s'interéssent aux progrès de la Mécanique céleste...doivent éprouver quelque étonnement en voyant combien de fois on a démontré la stabilité du système solaire.
Lagrange l'a établie d'abord, Poisson l'a démontrée de nouveau, d'autres démonstrations sont venues depuis, d'autres viendront encore. Les démonstrations anciennes étaient-elles insuffisantes, ou sont-ce les nouvelles qui sont superflues?

Those who are interested in the progress of celestial mechanics…must feel some astonishment at seeing how many times the stability of the Solar System has been demonstrated.
Lagrange established it first, Poisson has demonstrated it again, other demonstrations came afterwards, others will come again. Were the old demonstrations insufficient, or are the new ones unnecessary?

Poincaré (1897)

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- only feasible approach is numerical computation of the planetary orbits

Long-term numerical integrations of the solar system

why are these hard?

- most improvements in speed in modern computers come through massive parallelization, and this problem is difficult to parallelize
  - for N planets only N(N-1)/2 operations can be done in parallel; if N=8 then N(N-1)/2=28
  - parallel-in-time (e.g., parareal) algorithms have not been explored much (Saha, Stadel, & Tremaine 1997, Jiménez-Pérez & Laskar 2011)

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- sophisticated integration algorithms are needed to avoid numerical dissipation

Consider following a test particle in the force field of a point mass. Set G=M=1 for simplicity. Equations of motion read

$$\dot{\mathbf{r}} = \mathbf{v} \quad ; \quad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}) = -\frac{\mathbf{r}}{r^3}$$

Examine three integration methods with timestep h:

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}_n)$$

1. Euler's method

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}_{n+1})$$

2. modified Euler's

$$\mathbf{r}' = \mathbf{r}_n + \frac{h}{2}\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}') \quad ; \quad \mathbf{r}_{n+1} = \mathbf{r}' + \frac{h}{2}\mathbf{v}_{n+1}$$

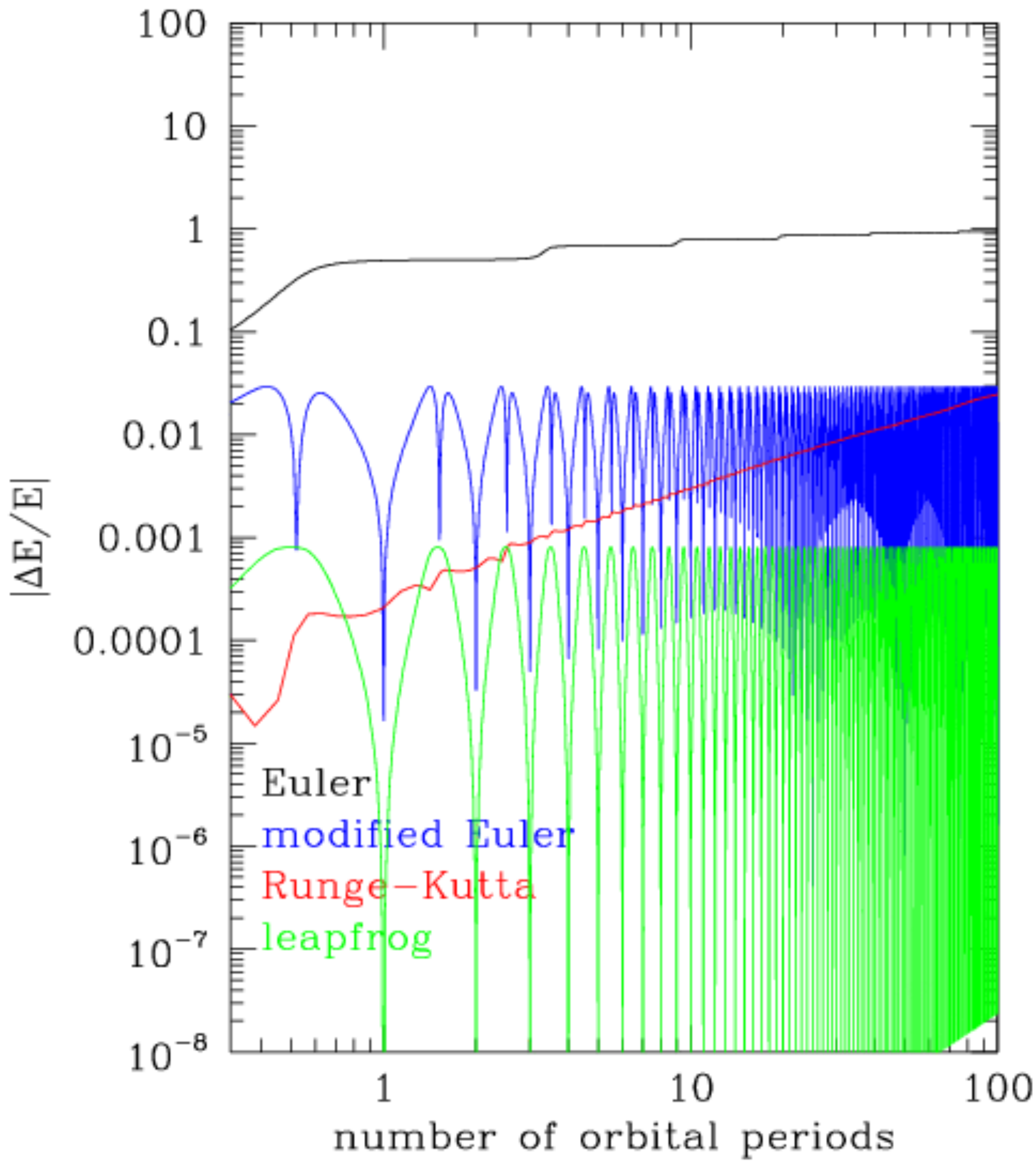
3. leapfrog

4. Runge-Kutta method

Euler methods are first-order; leapfrog is second-order; Runge-Kutta is fourth order eccentricity = 0.2

200 force evaluations per orbit with each method

plot shows fractional energy error |ΔE/E|



Motion of a test particle in a potential $\Phi(\mathbf{r})$ is described by the Hamiltonian

$$H(\mathbf{r},\mathbf{v}) = H_A(\mathbf{r},\mathbf{v}) + H_B(\mathbf{r},\mathbf{v})$$

where

$$H_A = \frac{1}{2}v^2 \quad ; \quad H_B = \Phi(\mathbf{r})$$

and the equations of motion

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{v}} \quad ; \quad \dot{\mathbf{v}} = -\frac{\partial H}{\partial \mathbf{r}}.$$

To create an integrator with time step h, advance the particle for h under H_A alone and then for h under H_B alone: (operator splitting)

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n - \frac{\partial \Phi}{\partial \mathbf{r}}(\mathbf{r}_{n+1})$$

which is modified Euler.

Modified Euler is a symplectic or Hamiltonian map because at each step the particle trajectory is determined by a Hamiltonian.

Motion of a test particle in a potential $\Phi(\mathbf{r})$ is described by the Hamiltonian

whe

and

A *geometric integration algorithm* is a numerical integration algorithm that preserves some geometric property of the original set of differential equations (e.g., symplectic algorithms, time-reversible algorithms)

The motivation for geometric integration algorithms is that preserving the phase-space geometry of the flow determined by the real dynamical system is more important than minimizing the one-step error

To

alon

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$$\Phi(\mathbf{r},t) = -\frac{GM}{\mathbf{r}} - \sum_{j=1}^{N} \frac{Gm_j}{|\mathbf{r}-\mathbf{r}_j|}.$$

In this case a much better split is

$$H_A = \frac{1}{2}v^2 - \frac{GM}{\mathbf{r}} \quad ; \quad H_B = -\sum_{j=1}^{N} \frac{Gm_j}{|\mathbf{r}-\mathbf{r}_j|}$$

because the integration errors are smaller by $\mathrm{O}(m/M)$.

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mixed-variable symplectic integrator
(Wisdom & Holman 1992)

Long-term numerical integrations of the solar system

why are these hard?

- most improvements in speed in modern computers come through massive parallelization, and this problem is difficult to parallelize
- sophisticated algorithms are needed to avoid numerical dissipation
- roundoff error:
  - typically a few bits per timestep ⇒ fractional error of a few times 2^{-53} in standard double precision ~ a few times 10^{-16}
  - <u>systematic roundoff</u>: 20 steps/orbit × 10^{10} orbits × 2^{-53} (53 bits in double precision) = 2×10^{-5}
  - <u>random roundoff</u>: (20 steps/orbit × 10^{10} orbits)$^{1/2}$ × 2^{-53} = 5×10^{-11}
  - how to eliminate systematic roundoff:
    - use machines with optimal floating-point arithmetic (IEEE 754 standard)
    - eliminate all fixed non-representable numbers (⅓, π, etc.)
    - check that errors in orbital elements grow as $t^{1/2}$, not t

The equations of motion for the solar system

Newton's law of gravity and Newton's laws of motion for 8 planets + the Sun:

$$\frac{d^2\mathbf{x}_i}{dt^2} = G\sum_{j\neq i}\frac{m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}(\mathbf{x}_j - \mathbf{x}_i) \qquad + \text{small corrections}$$

"Small corrections" include:

- satellites of the planets
- general relativity
- largest asteroids

All are at levels of less than 10^{-6} and all are straightforward to include

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Unknowns include:

- smaller asteroids and Kuiper belt beyond Neptune
- mass loss from Sun
- drag of solar wind on planetary magnetospheres
- tidal forces from the Milky Way
- passing stars (highly unlikely)
- errors in planetary masses or initial conditions

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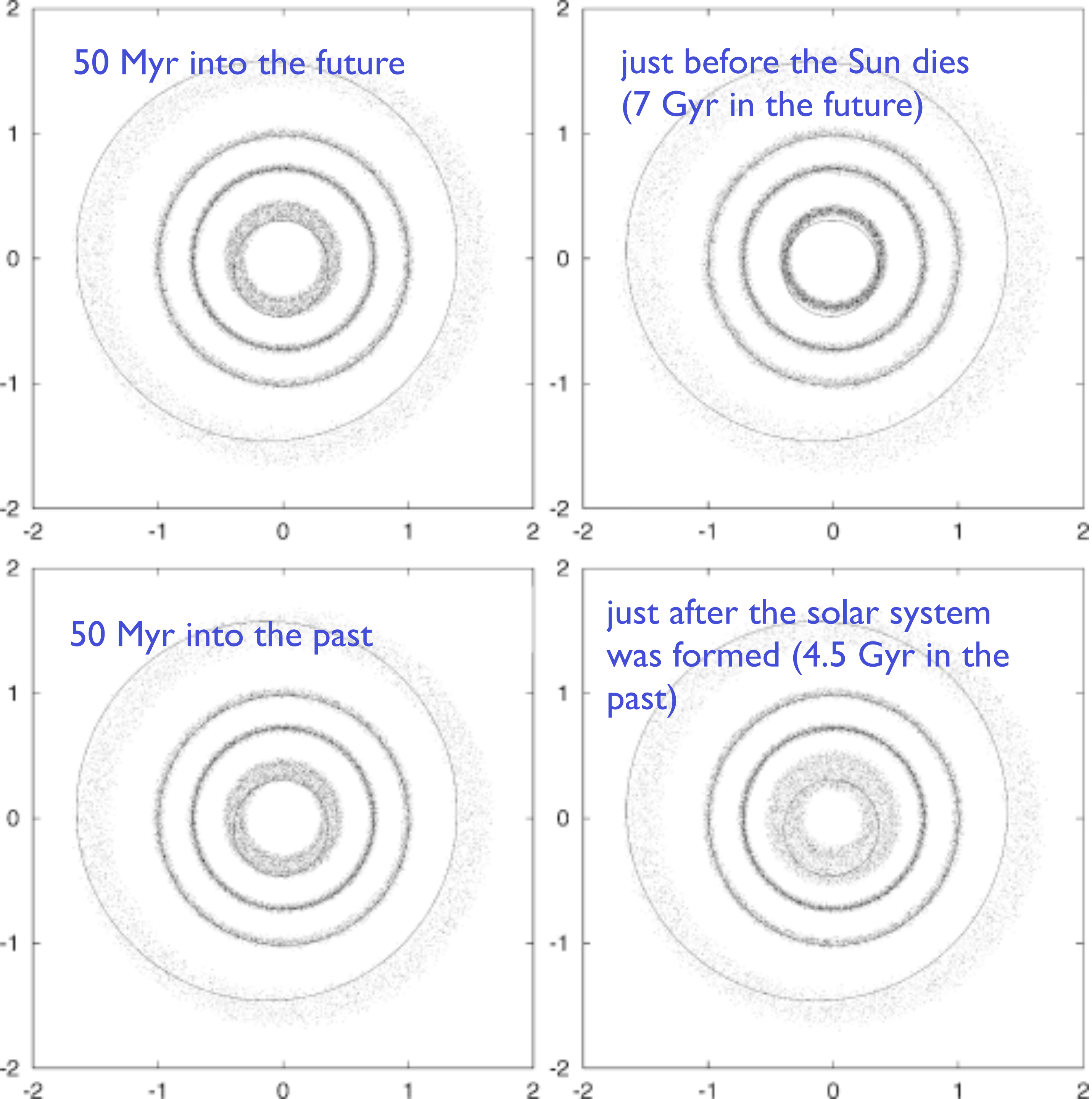


innermost four planets (Mercury, Venus, Earth, Mars)

Ito & Tanikawa (2002)

Two kinds of dynamical system

Regular

- highly predictable, "well-behaved"
- small differences grow linearly: $\Delta x, \Delta v \propto t$
- e.g. baseball, golf, simple pendulum, all problems in mechanics textbooks, planetary orbits on short timescales

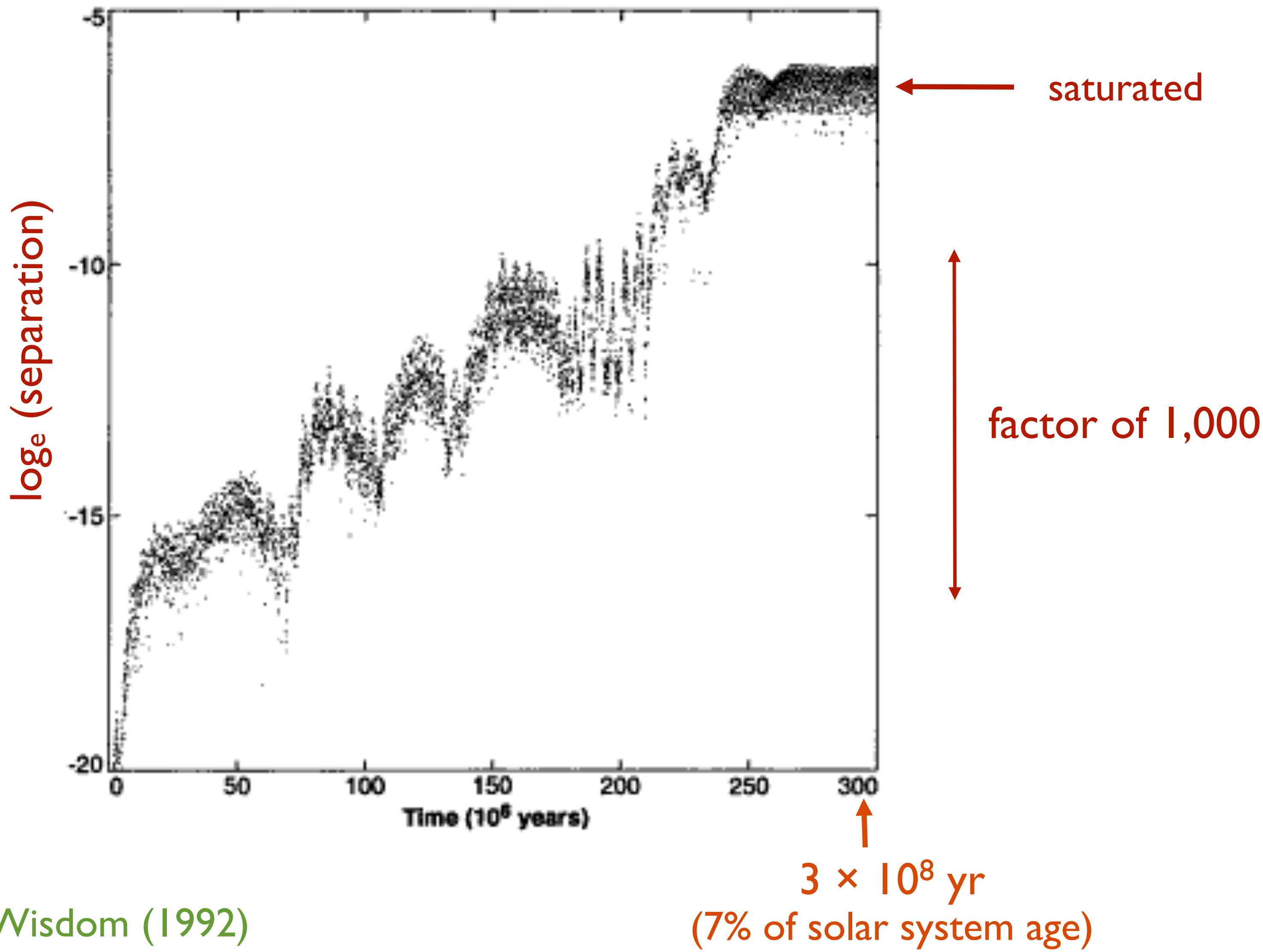
Chaotic

- difficult to predict, "erratic"
- small differences grow exponentially at large times: $\Delta x, \Delta v \propto \exp(t/t_L)$ where t_L is Liapunov time
- appears regular on timescales short compared to Liapunov time $\Rightarrow$ linear growth of small changes on short times, exponential growth on long times
- e.g. roulette, dice, pinball, weather, billiards, double pendulum

The stability of the solar system

- all planetary orbits are chaotic, with Liapunov time $t_L \sim$ 5-20 Myr $\Rightarrow$ > 200 e-folds in the lifetime of the solar system (Sussman & Wisdom 1988, Laskar 1989, Sussman & Wisdom 1992, Hayes et al. 2010)

Jupiter



Sussman & Wisdom (1992)

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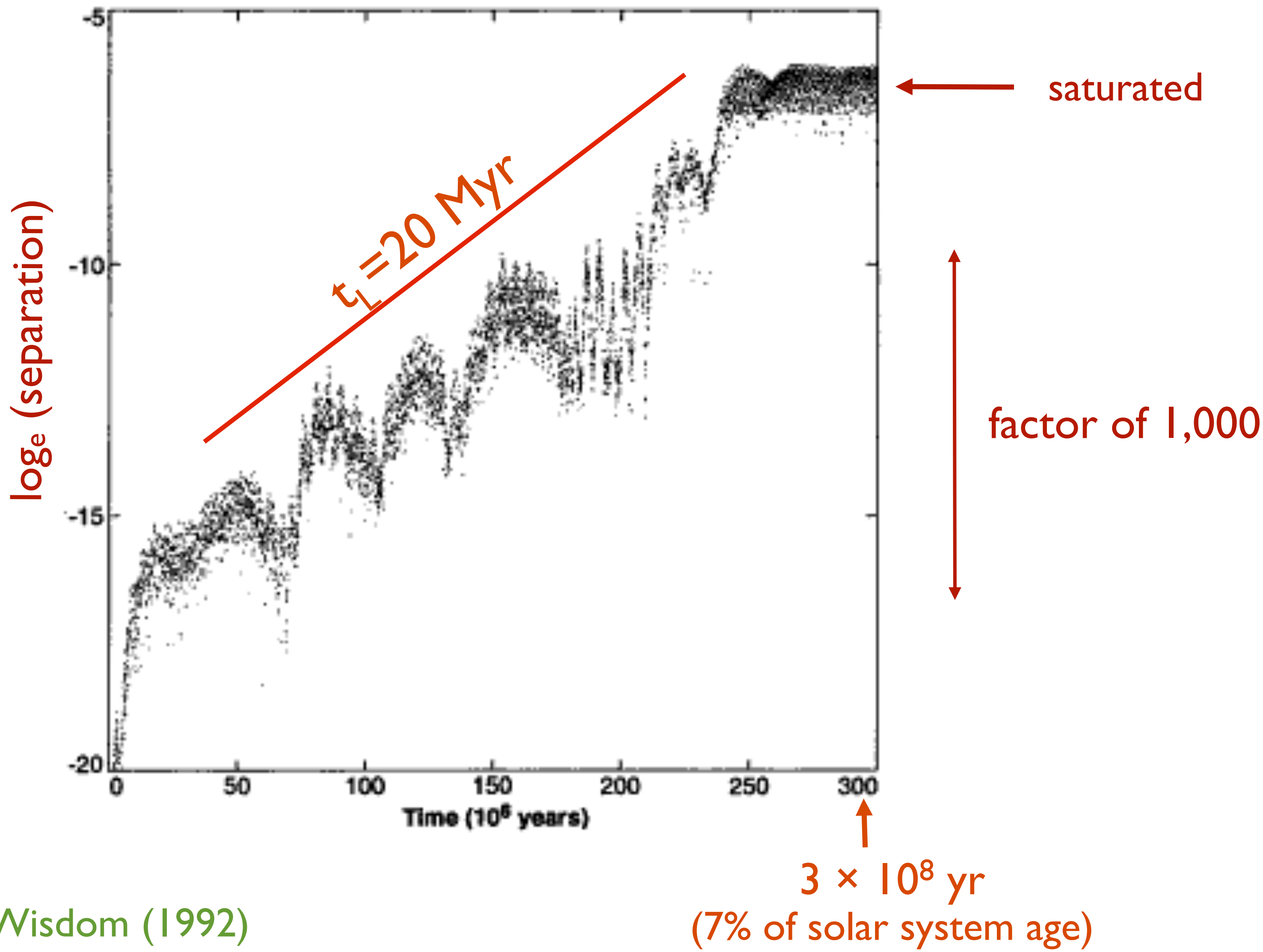

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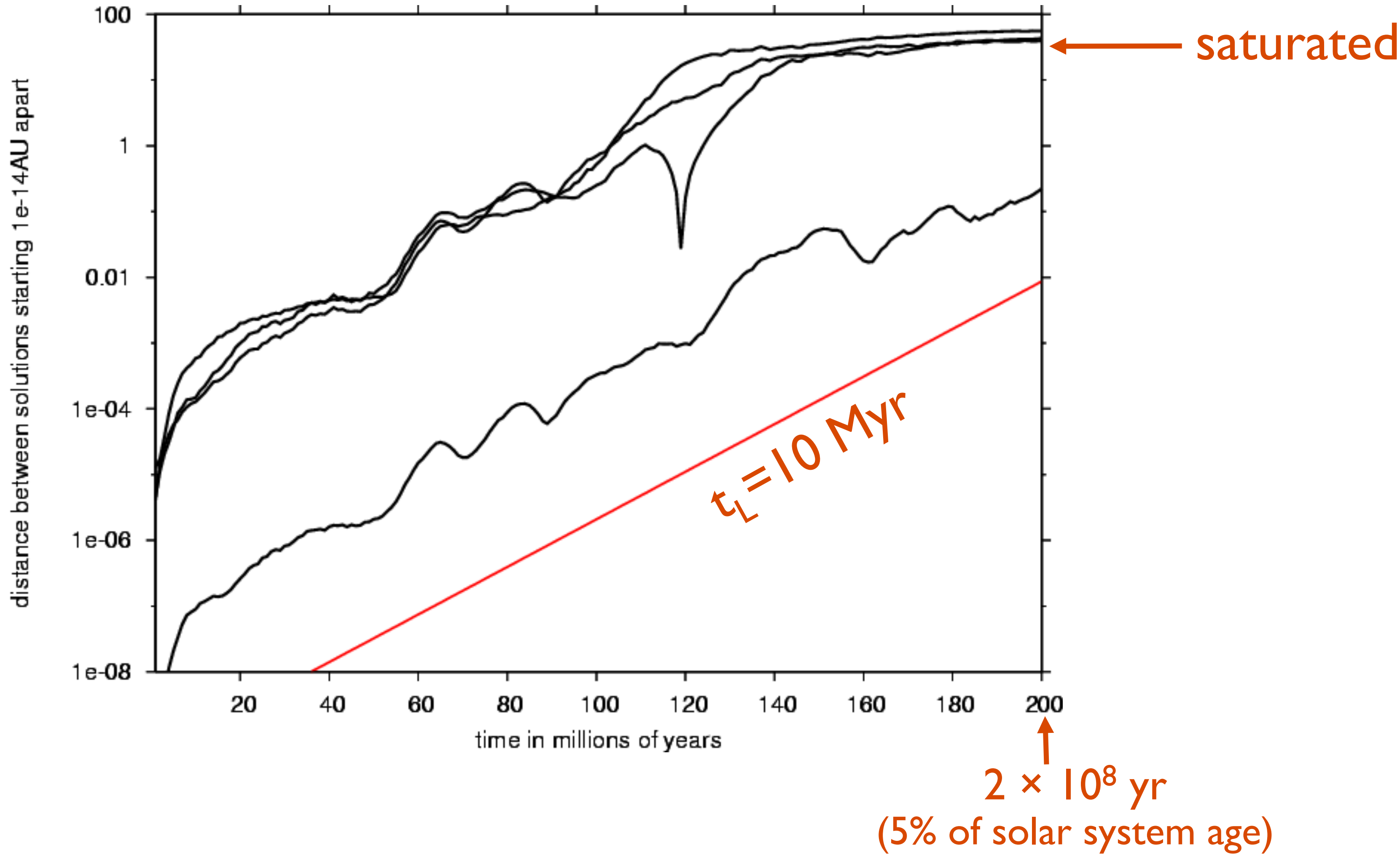

- double-precision (p=53 bits) 2nd order mixed-variable symplectic method with h=4 days and h=8 days
- double-precision (p=53 bits) 14th order multistep method with h=4 days
- extended-precision (p=80 bits) 27th order Taylor series with h=220 days

Hayes (2008)

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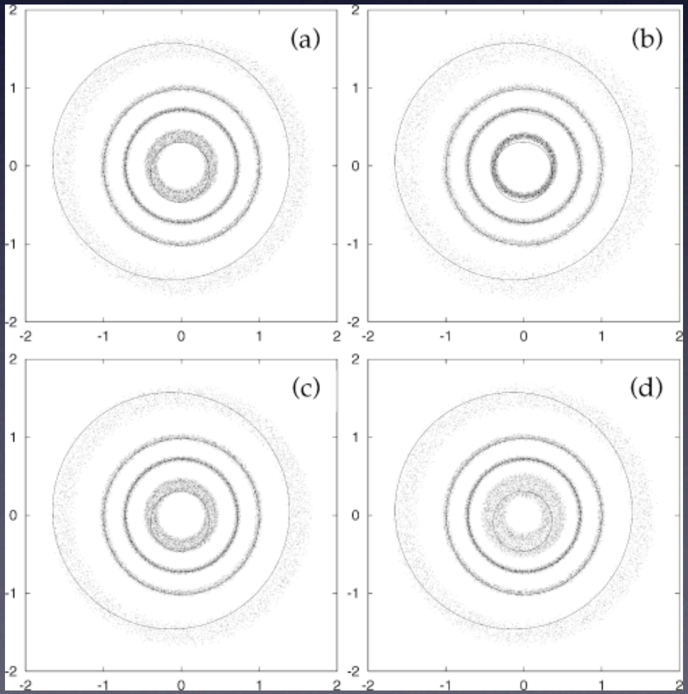

Ito & Tanikawa (2002)

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- most of the chaotic behavior is in the orbital phases of the planets, not the overall shapes and sizes of the orbits
- implications:
    - accurate predictions for the positions of the planets can only be made for ~1% of the age of the solar system
    - for longer times we can only make statistical statements about the future of the solar system, by running many calculations with small changes in initial conditions
    - solar system is a bad example of a clockwork universe

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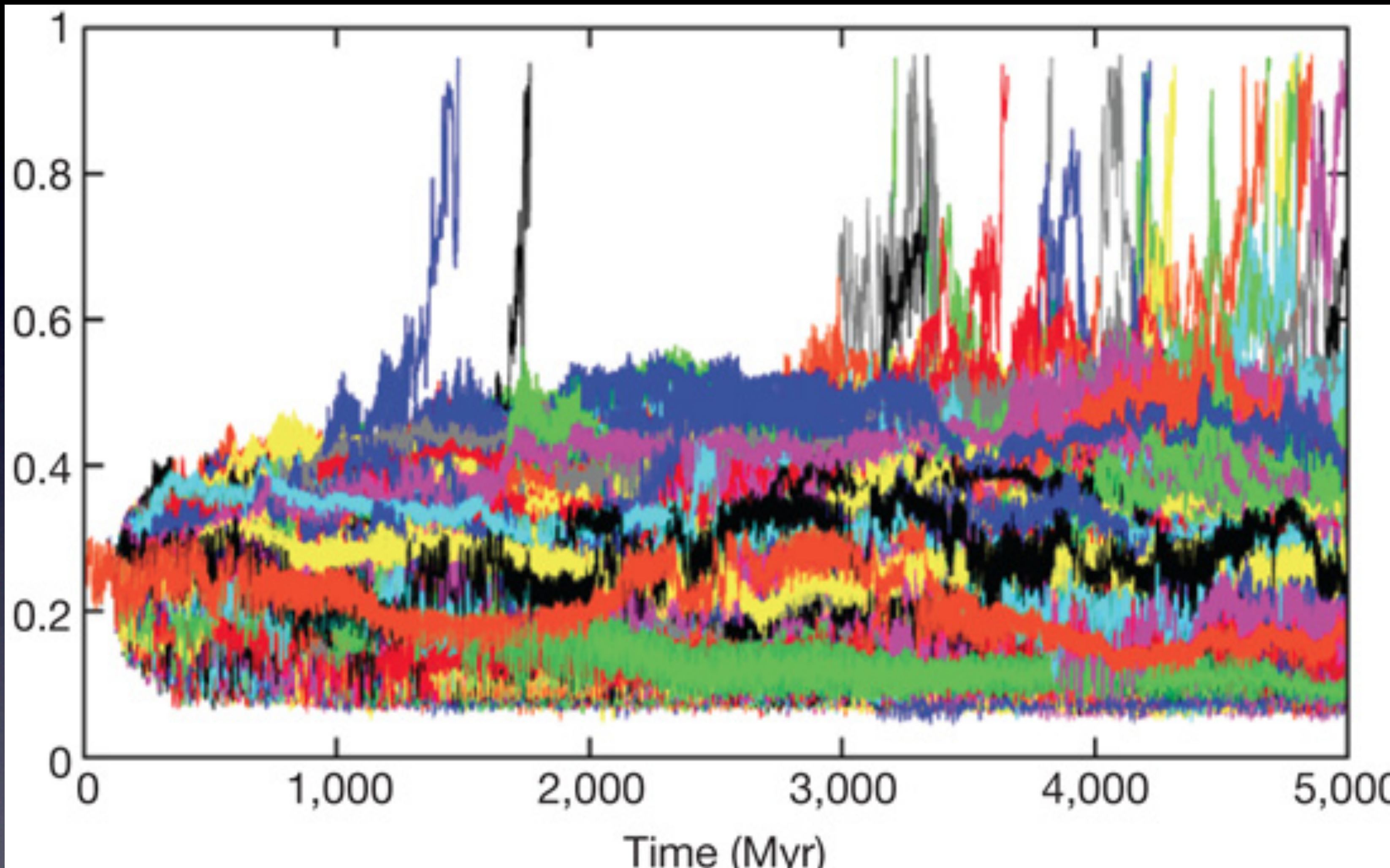
- however, the shape of Mercury's orbit changes randomly
- in about 1% of integrations, Mercury undergoes a catastrophic event (collision with Sun or another planet, escape from the solar system, etc.)

maximum eccentricity of Mercury over 1 Myr running window, for 2500 nearby initial conditions



Laskar & Gastineau (2009)

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- in about 1% of integrations, Mercury undergoes a catastrophic event (collision with Sun or another planet, escape from the solar system, etc.)
- results are very sensitive to details:
  - not including relativity increases fraction of high-eccentricity outcomes from 1% to 60%
  - even within observational error in initial conditions, only ~70% of trajectories are chaotic (Hayes 2008)

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- however, the shape of Mercury's orbit changes randomly
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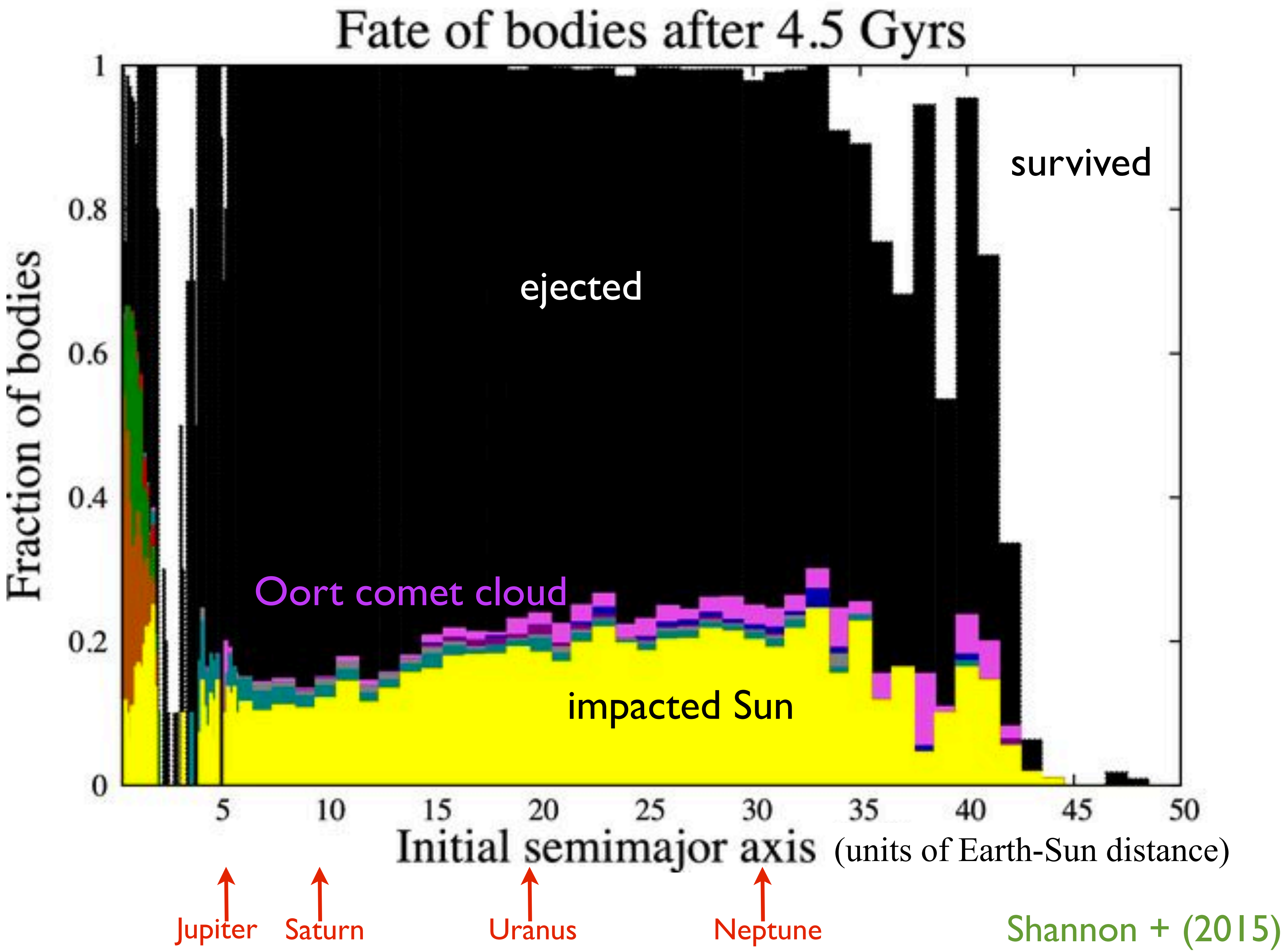
- results are very sensitive to details
- most likely, ejections or collisions of planets have already occurred



Fate of bodies after 4.5 Gyrs
Fraction of bodies
0
0.2
0.4
0.6
0.8
1
5
10
15
20
25
30
35
40
45
50
Initial semimajor axis (units of Earth-Sun distance)
survived
ejected
Oort comet cloud
impacted Sun
Jupiter
Saturn
Uranus
Neptune
Shannon + (2015)

- orbits of planets in the solar system are chaotic
- probably chaotic evolution of orbits has led to collisions and ejections of planets in the past
- can aspects of this process be described analytically i.e., without integrating orbits?

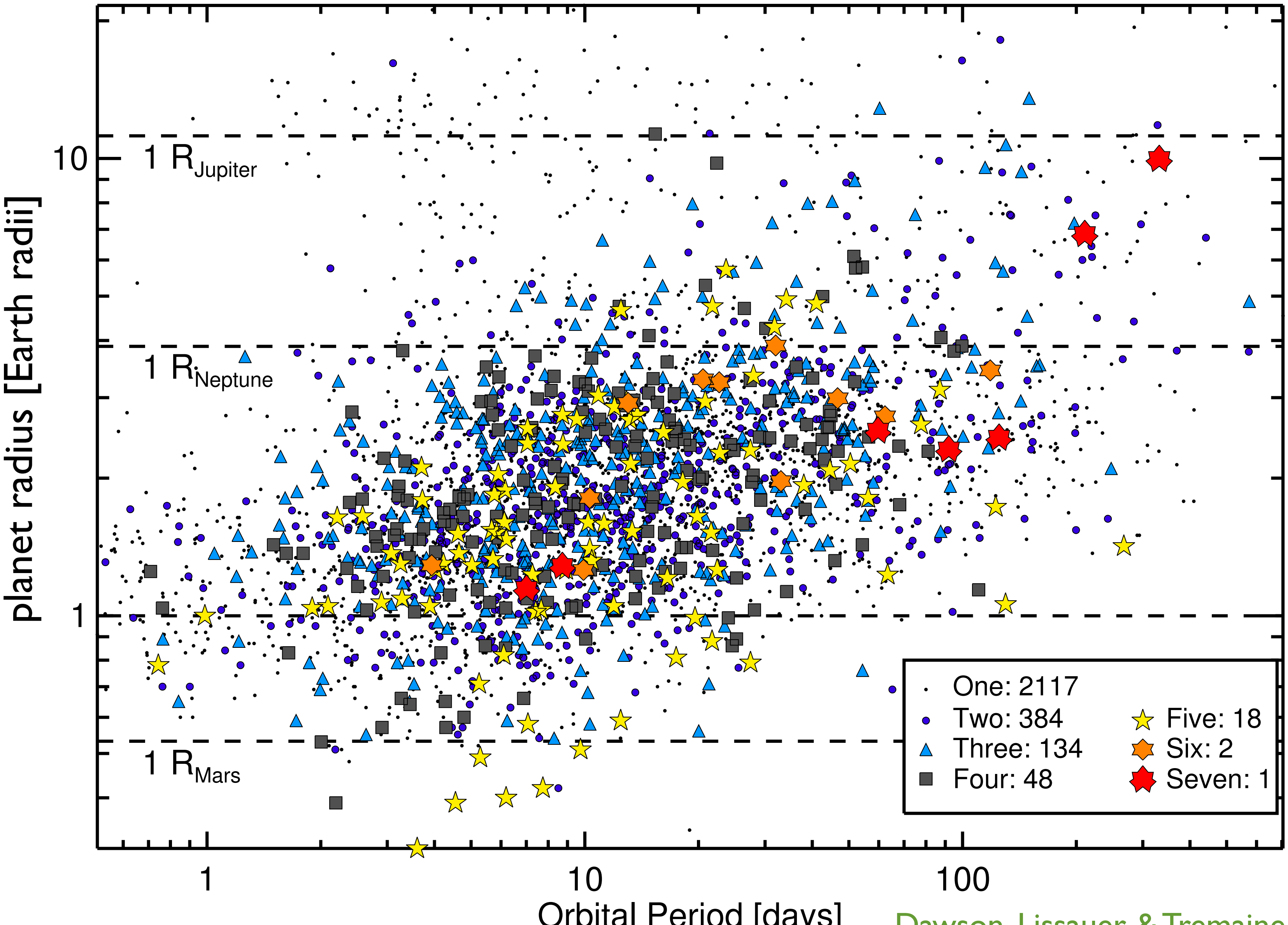
There are many bad examples of attempts to explain the properties of planetary orbits from first principles, e.g.,

- Kepler's zeroth law
- Titius-Bode law

Nevertheless there are reasons to try again:

- N-body integrations allow approximate analytic models to be tested
- *Kepler* has provided a large statistical sample of multi-planet systems

Planetary systems discovered by *Kepler*



Dawson, Lissauer, & Tremaine (2015)

The range of strong interactions from a planet of mass m orbiting a star of mass M in a circular orbit of radius a is the Hill radius

$$r_H = a\left(\frac{m}{3M}\right)^{1/3}.$$

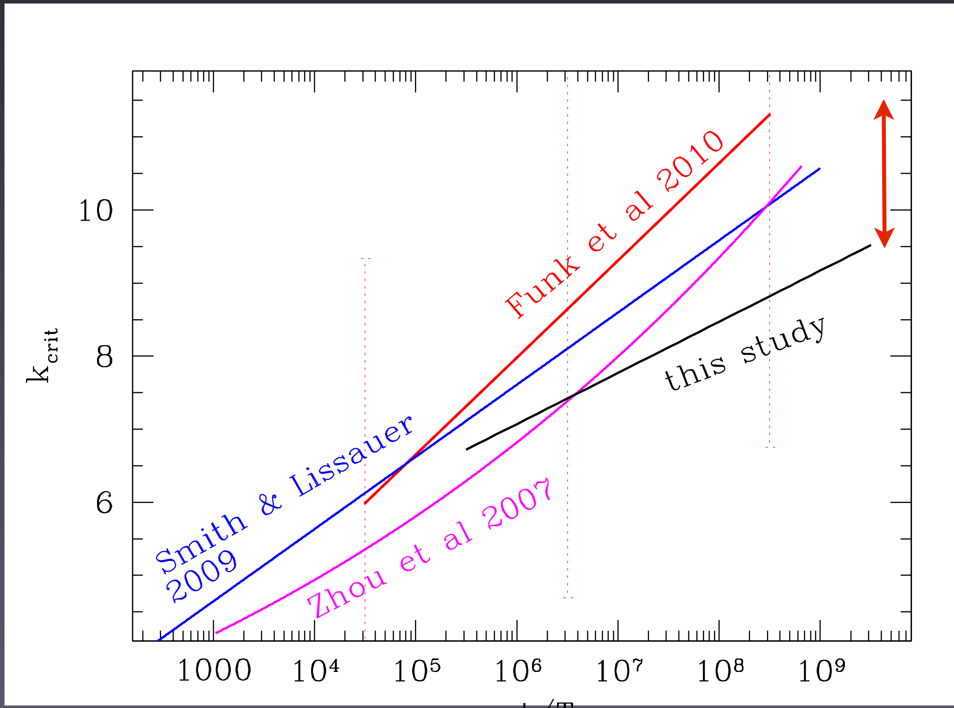
Numerical integrations show that planets of mass m, m′ with semi-major axes a, a’, a < a’ are stable for N orbital periods if closest approach exceeds k Hill radii, or

$$a'(1-e') - a(1+e) > k(N) r_H$$

pericenter of outer planet

apocenter of inner planet

typically $k(10^{10}) \simeq 11 \pm 2$



Pu & Wu (2014)

The sheared sheet

Problem: statistical mechanics works best on homogeneous systems with N >> 1, whereas planetary systems have large-scale radial gradients and N < 10

Equation of motion for a planet is

$$\ddot{\mathbf{R}} + \frac{GM_\star}{R^3}\mathbf{R} = -\nabla\Phi$$

where Φ is the gravitational potential from the other planets.

Transform to frame rotating with angular speed $\Omega = (GM_\star/R_0^3)^{1/2}$ appropriate for a circular orbit at radius R_0:

$$\ddot{\mathbf{R}} + \frac{GM_\star}{R^3}\mathbf{R} - \Omega^2\mathbf{R} + 2\boldsymbol{\Omega}\times\mathbf{R} = -\nabla\Phi$$

Write $\mathbf{R} = \mathbf{R}_0 + x\hat{\mathbf{r}} + y\hat{\boldsymbol{\phi}}$ and expand to $\mathrm{O}(x, y/R_0)$:

$$\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x = -\frac{\partial\Phi}{\partial x}, \qquad \ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Phi}{\partial y}.$$

These are invariant under $x \to x + \Delta$, $y \to y + \frac{3}{2}\Omega\Delta t$ so we can apply periodic shearing boundary conditions.

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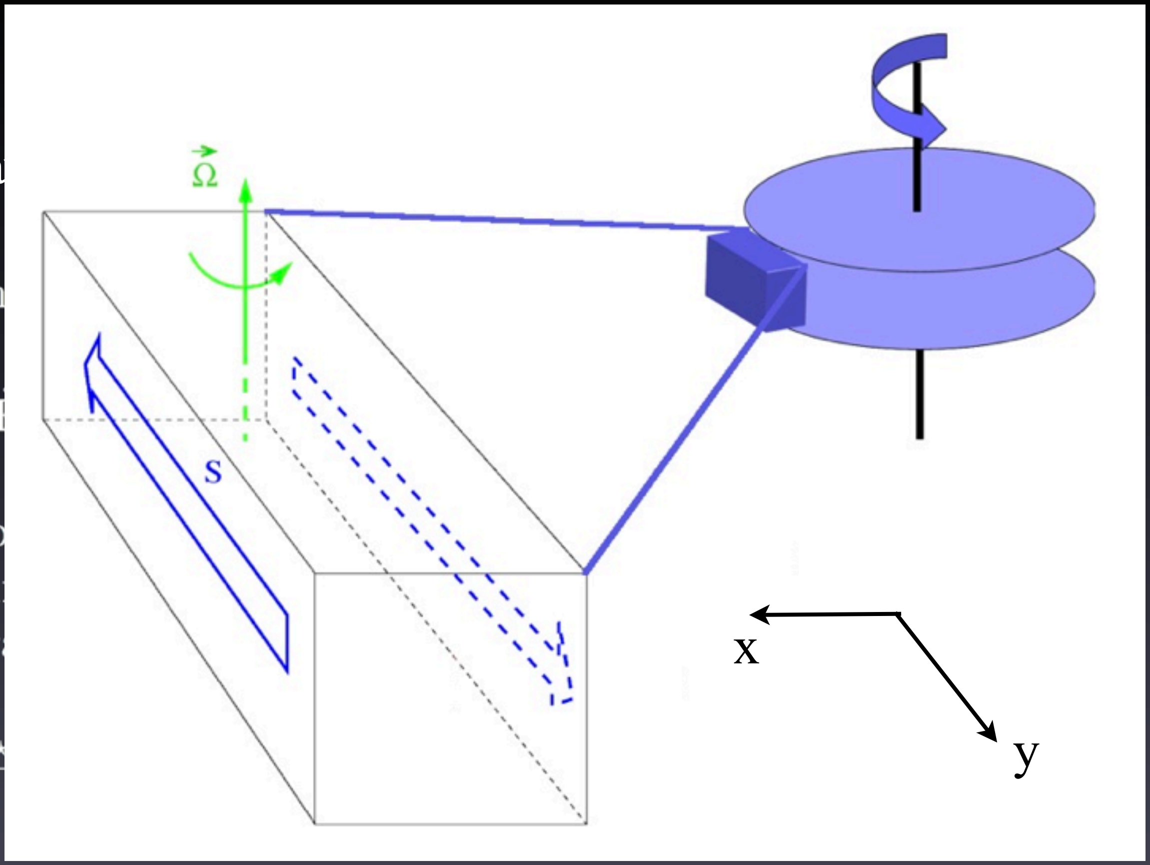
whereas planetary systems ha

Equation of motion for a plan

where Φ is the gravitational p
Transform to frame rotati
priate for a circular orbit at r

$$\ddot{\mathbf{R}} + \frac{GM_\star}{R^3}$$



Write $\mathbf{R} = \mathbf{R}_0 + x\hat{\mathbf{r}} + y\hat{\boldsymbol{\phi}}$ and expand to $\mathrm{O}(x, y/R_0)$:

$$\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x = -\frac{\partial\Phi}{\partial x}, \qquad \ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Phi}{\partial y}.$$

These are invariant under $x \to x + \Delta$, $y \to y + \frac{3}{2}\Omega\Delta t$ so we can apply periodic shearing boundary conditions.

The sheared sheet

Problem: statistical mechanics works best on homogeneous systems with N >> 1, whereas planetary systems have large-scale radial gradients and N < 10

Equation of motion for a planet is

$$\ddot{\mathbf{R}} + \frac{GM_\star}{R^3}\mathbf{R} = -\nabla\Phi$$

where Φ is the gravitational potential from the other planets.

Transform to frame rotating with angular speed $\Omega = (GM_\star/R_0^3)^{1/2}$ appropriate for a circular orbit at radius R_0:

$$\ddot{\mathbf{R}} + \frac{GM_\star}{R^3}\mathbf{R} - \Omega^2\mathbf{R} + 2\boldsymbol{\Omega}\times\mathbf{R} = -\nabla\Phi$$

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Tra ... o-

priate

Ansatz: planetary systems fill uniformly the region of phase space allowed by stability (~ ergodic model)

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These are invariant under $x \to x + \Delta$, $y \to y + \frac{3}{2}\Omega\Delta t$ so we can apply periodic shearing boundary conditions.

1. Use the sheared sheet approximation
2. Assume systems fill the region of phase space allowed by stability (ergodic model)

Leads to an N-planet distribution function

apocenter and pericenter must be separated by k Hill radii

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} \overbrace{da_i de_i^2}^{\text{phase-space volume}} \, H[a_{i+1}(1-e_{i+1}) - a_i(1+e_i) - k r_H]$$

step function

where $H(\cdot)$ is the step function, $k = 11 \pm 2$, and $r_H = \overline{a}(m_i + m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

For comparison the distribution function for a one-dimensional gas of hard rods of length L (Tonks 1936) is

$$p(a_1, \ldots, a_N) \propto \prod_{i=1}^{N} da_i \, H(a_{i+1} - a_i - L)$$

N-planet distribution function

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} da_i de_i^2\, H[a_{i+1}(1 - e_{i+1}) - a_i(1 + e_i) - kr_H]$$

where $H(\cdot)$ is the step function, $k = 11{\pm}2$, and $r_H = \overline{a}(m_i{+}m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

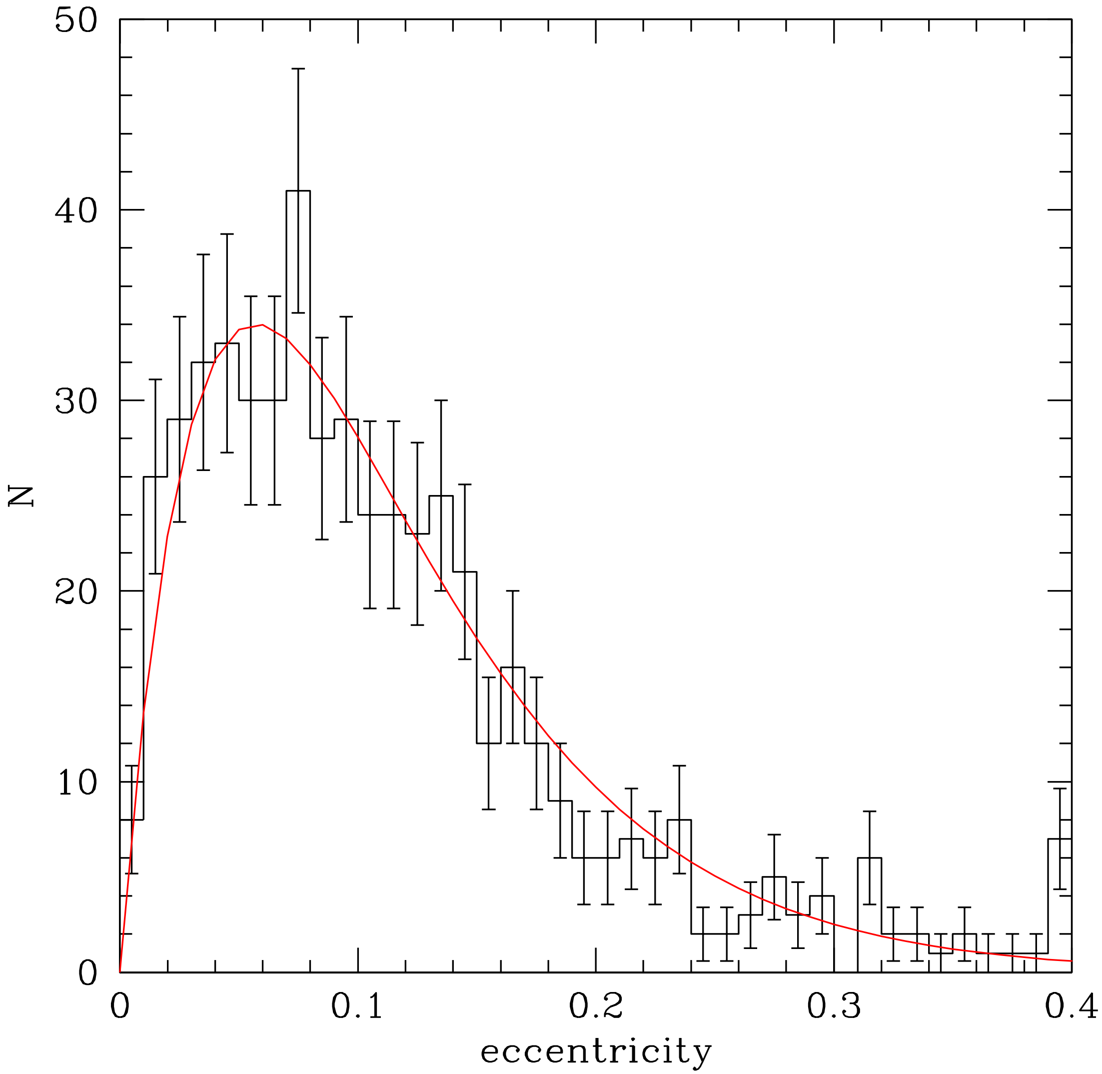
Predictions:

- eccentricity distribution:

$$p(e) = \int da \prod_{i=2}^{N} da_i de_i^2 p(a, e, \ldots, a_N, e_N) = \frac{e}{\tau^2} \exp\left(-\frac{e}{\tau}\right)$$

where τ is a free parameter

e.g., N-body simulations of planet growth by Hansen & Murray (2013)



$p(e) \sim e \exp(-e/\tau)$

$\tau = 0.060 \pm 0.003$

Statistical mechanics of planetary systems

N-planet distribution function

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} da_i de_i^2\, H[a_{i+1}(1-e_{i+1}) - a_i(1+e_i) - kr_H]$$

where $H(\cdot)$ is the step function, $k = 11 \pm 2$, and $r_H = \overline{a}(m_i + m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

Predictions:

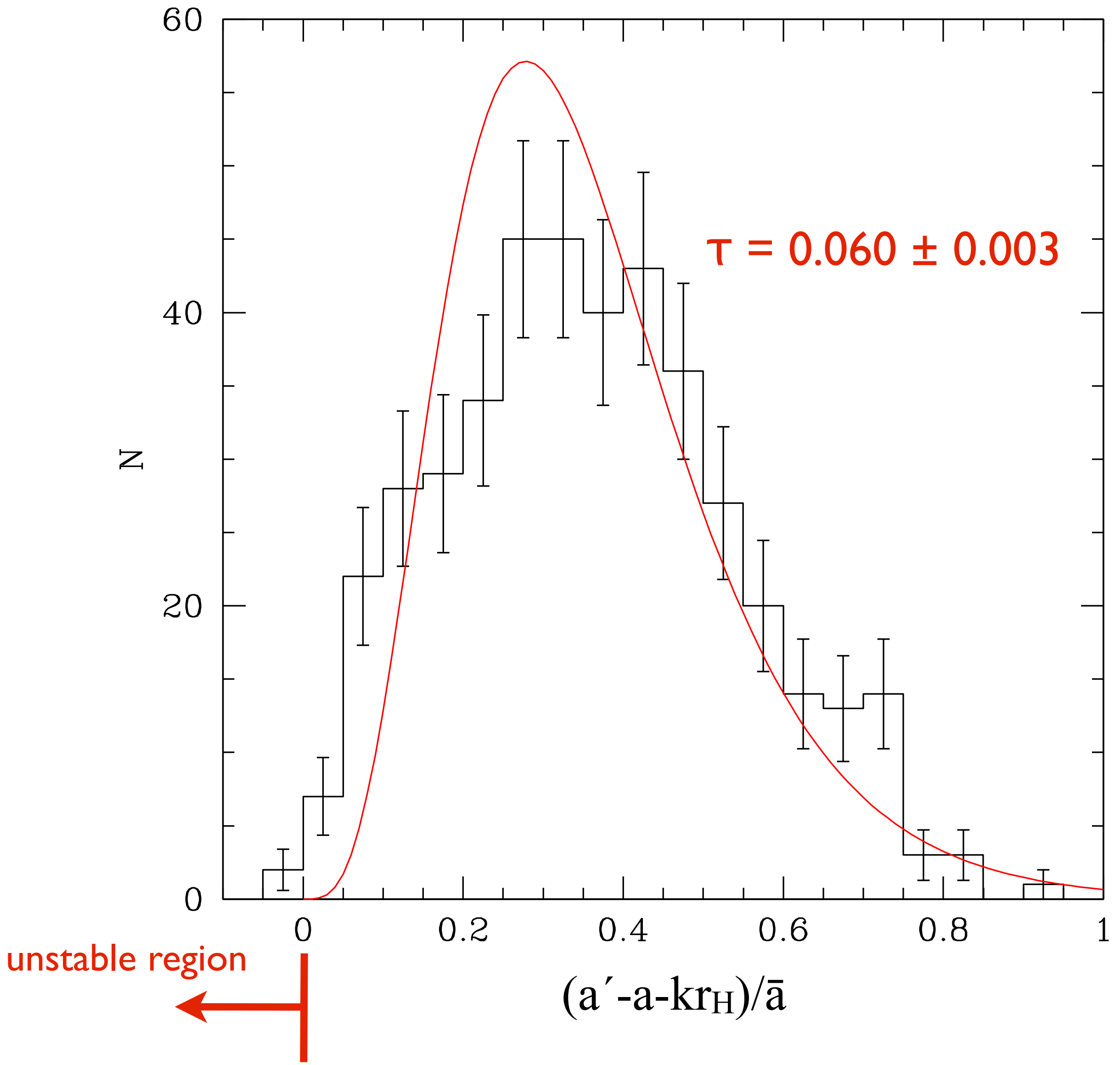
- eccentricity distribution ✔ with one free parameter
- distribution of semi-major axis differences between nearest neighbors:

$$p(a' - a) = \frac{4}{2\bar{a}\tau} G\left(\frac{a' - a - kr_H}{2\bar{a}\tau}\right)$$

where

$$G(x) = 6\exp(-x) - \exp(-2x)(x^3 + 3x^2 + 6x + 6).$$

e.g., N-body simulations of planet growth by Hansen & Murray (2013)



Statistical mechanics of planetary systems

N-planet distribution function

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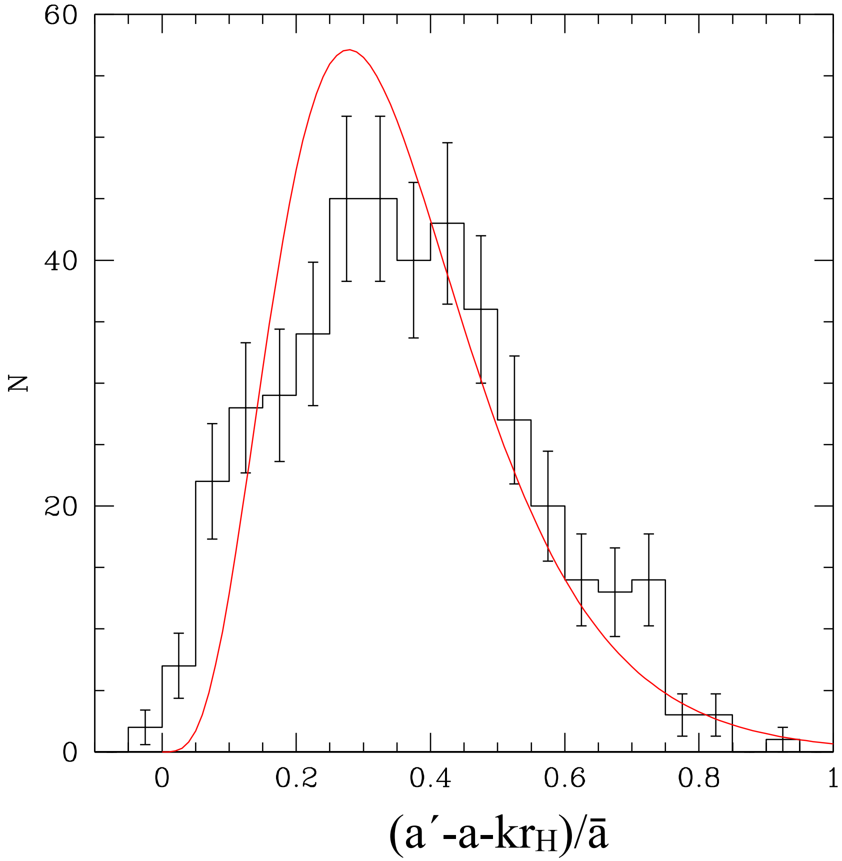
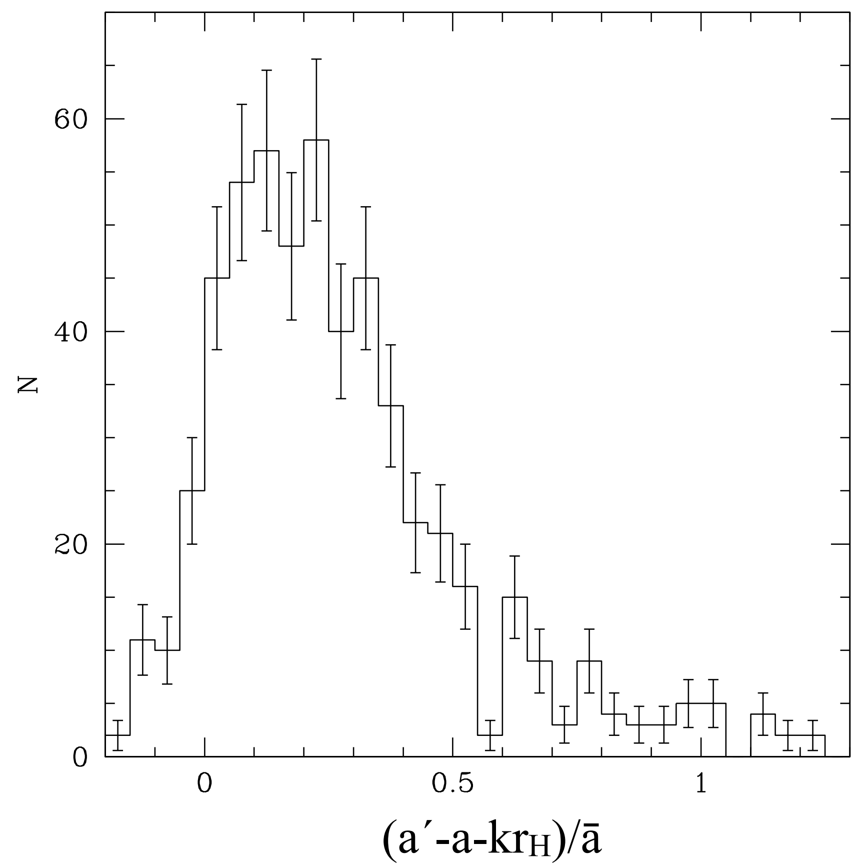
Predictions:

- eccentricity distribution ✔ with one free parameter
- distribution of semi-major axis differences ✔ with no free parameters

Hansen & Murray (2013) simulations



Kepler planets, using observed mass-radius relation (Weiss & Marcy 2014):



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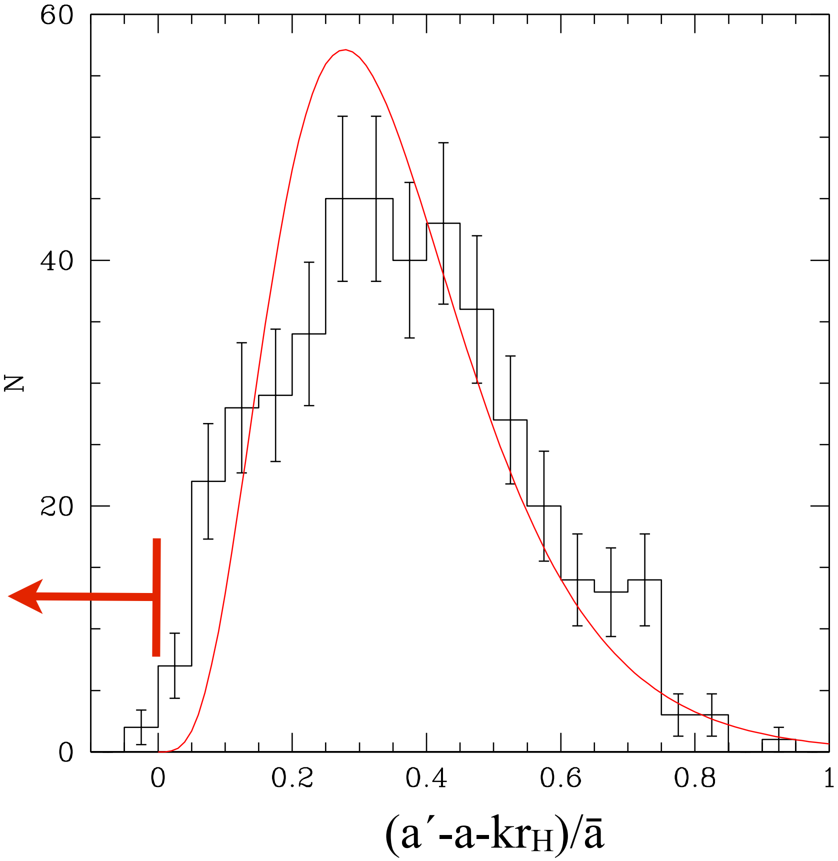

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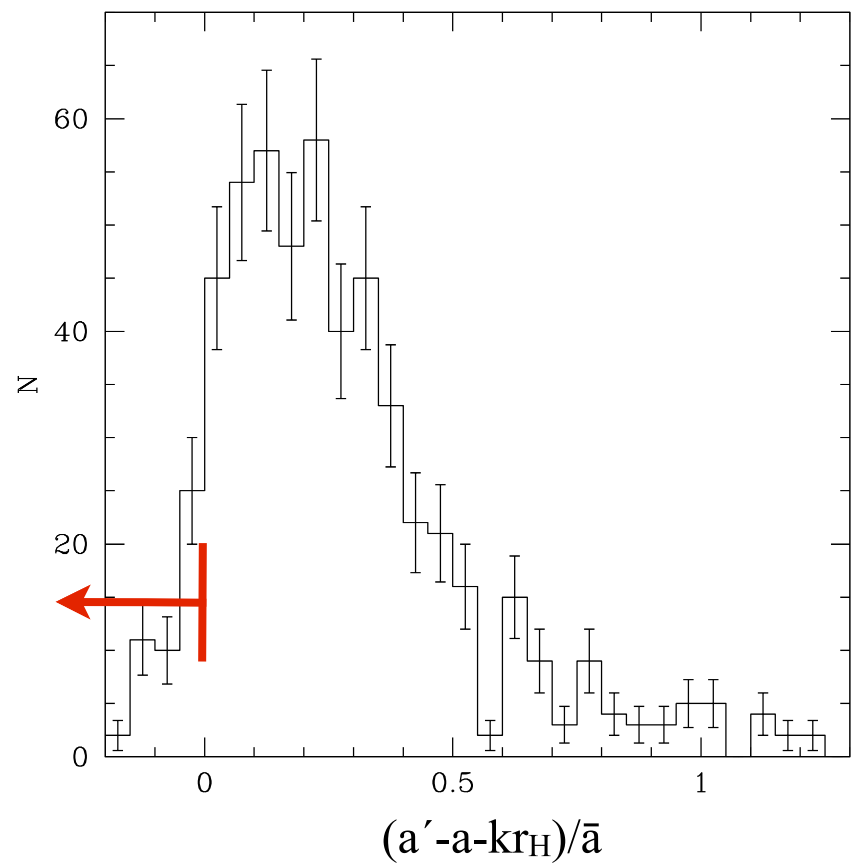

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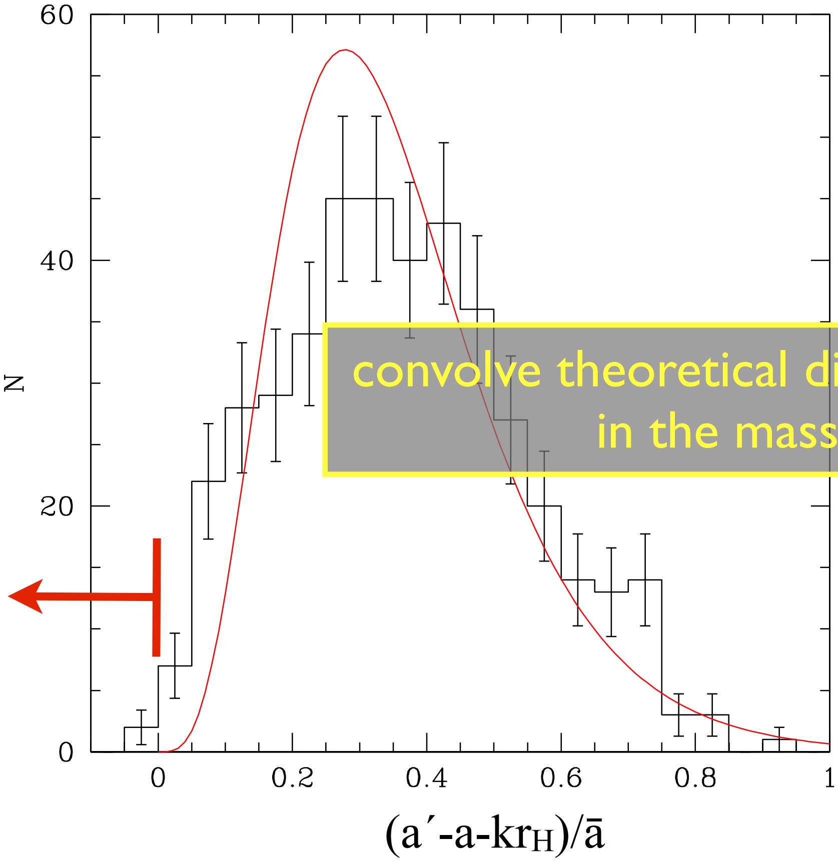

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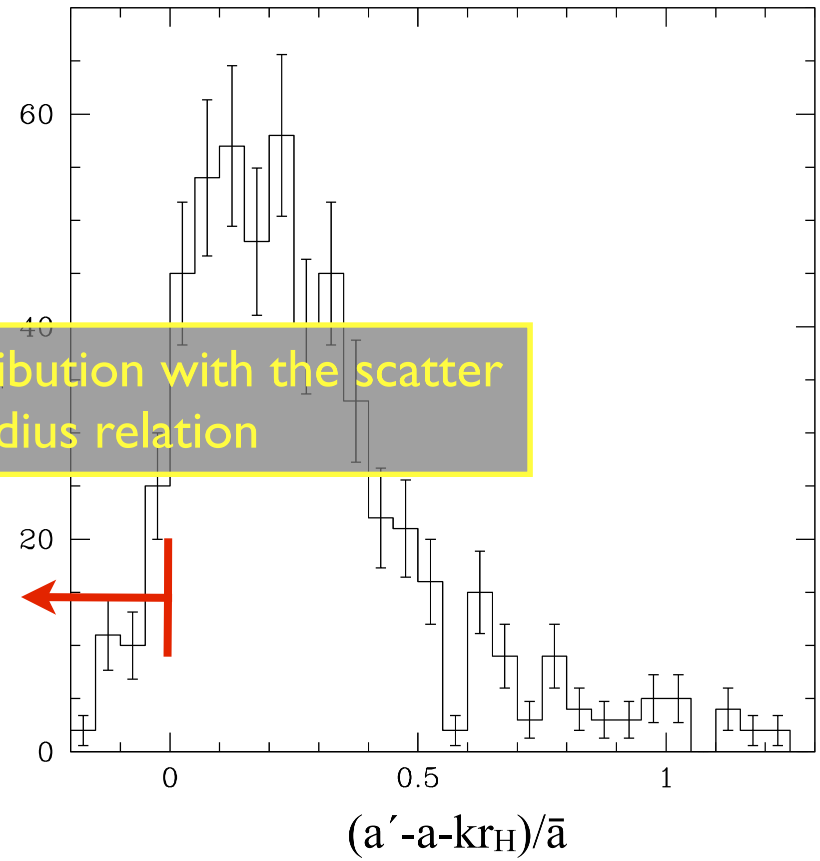

convolve theoretical distribution with the scatter in the mass-radius relation

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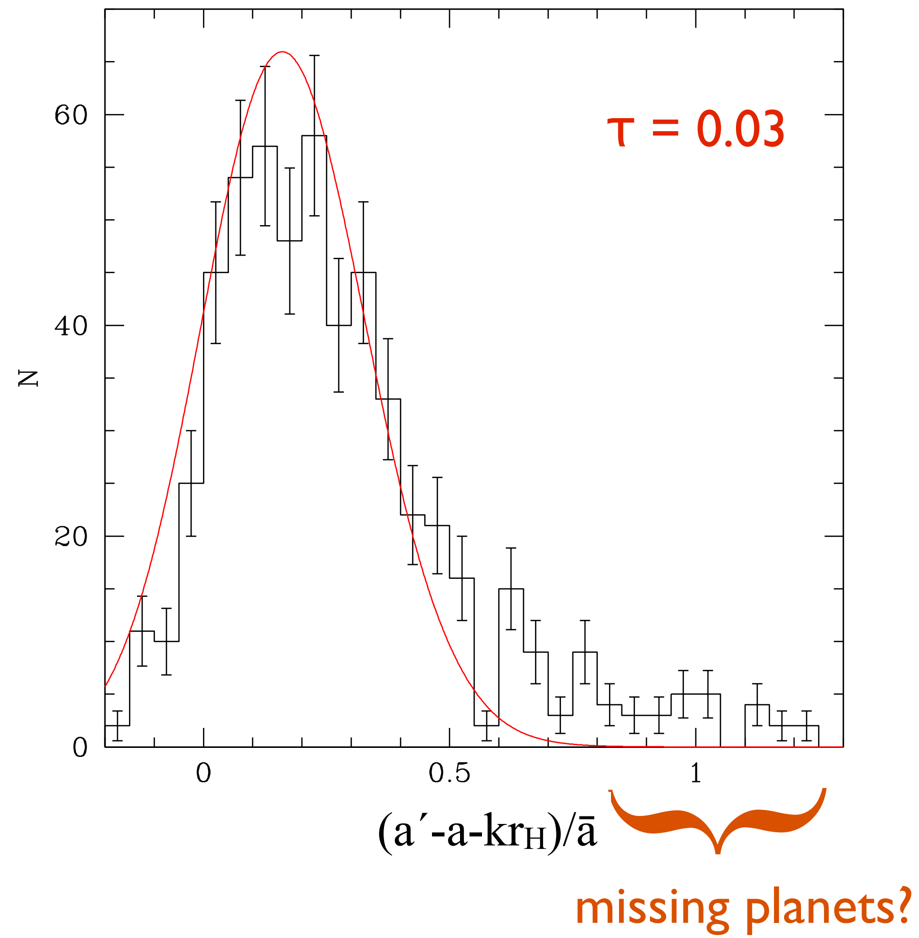

- with τ=0.03 ergodic model predicts <e>=0.06
  - <e>≃0.02-0.03 (Hadden & Lithwick 2014,2015)
  - <e>≃0.03 (Fabrycky et al. 2014)
  - <e>≃0.04 (van Eylen & Albrecht 2015)
  - <e>≃0.07 (Shabram et al. 2015)

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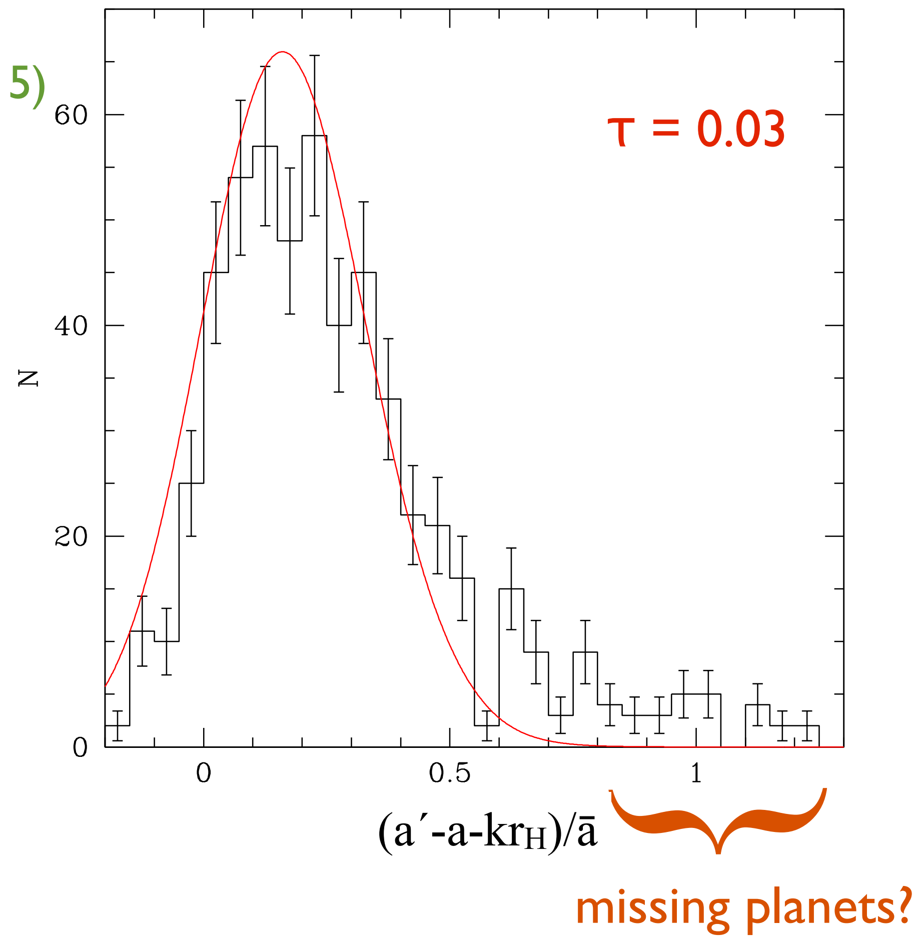

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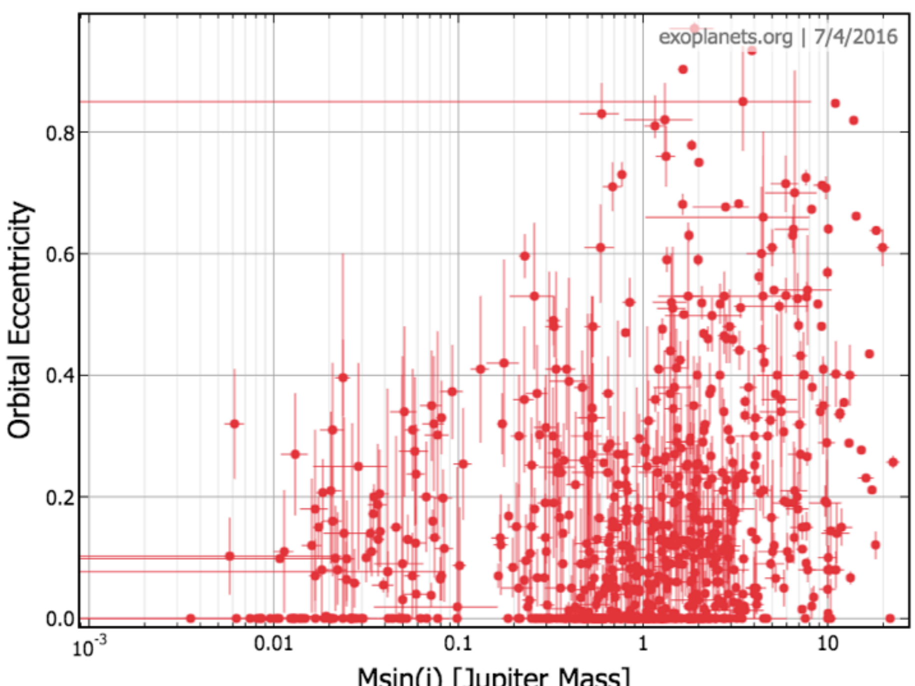
- ergodic model predicts no correlation between mass and eccentricity in a given system



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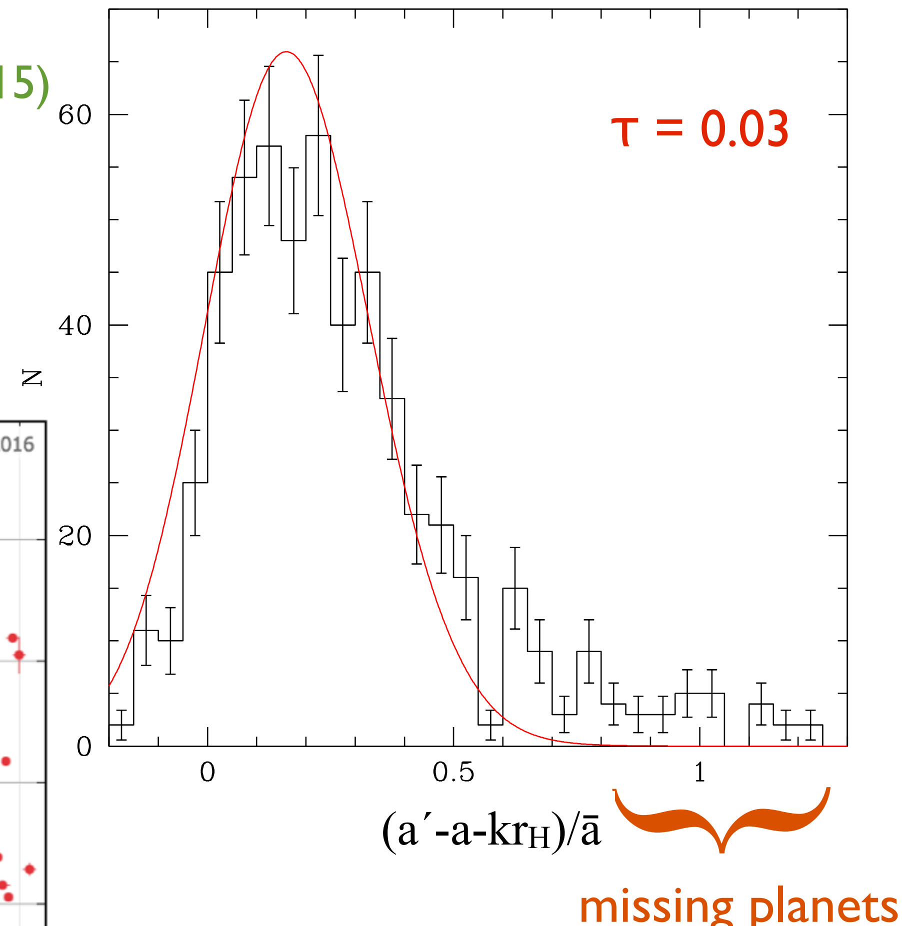

The stability of the solar system

- all planetary orbits are chaotic, with Liapunov time $t_L \sim$ 5-20 Myr $\Rightarrow$ > 200 e-folds in the lifetime of the solar system
- most of the chaotic behavior is in the orbital phases of the planets, not the overall shapes and sizes of the orbits;
- however, the eccentricity of Mercury's orbit undergoes a random walk and there is about a 1% chance that it will be destroyed before the end of the Sun's life
- results are very sensitive to details, e.g., relativistic effects
- most likely, ejections or collisions of planets have already occurred
- simple ergodic models capture many of the statistical properties of the orbits in extrasolar planetary systems